The triple wave T_2 is bounded by the planes

$$\tfrac{1}{2}\sqrt{3}(\gamma - 1)\,\xi_2 - \tfrac{1}{2}(\gamma + 1)\,\xi_3 + \tfrac{1}{2}(\gamma - 1)(M_0 - V) = 0$$
$$\tfrac{1}{2}(7\gamma - 1)\,\xi_1 + \tfrac{1}{2}\sqrt{3}(\gamma + 1)\,\xi_2 - (\gamma - 1)(\xi_3 + M_0) + \tfrac{3}{2}(3\gamma - 1)V = 0$$
$$\tfrac{1}{2}(7\gamma - 1)\,\xi_1 - \tfrac{1}{2}\sqrt{3}(\gamma + 1)\,\xi_2 + (\gamma - 1)(\xi_3 + M_0 - V) = 0$$
$$\tfrac{1}{2}\sqrt{3}(\gamma - 1)\,\xi_2 - \tfrac{1}{3}(\gamma + 1)\,\xi_3 + \tfrac{1}{2}(\gamma - 1)(M_0 - \tfrac{10}{3}V) = 0$$
$$\xi_1 = -V \quad \text{(piston)}$$

All of the side faces of the regions in the lower half of Fig. 2 are orthogonal to the piston $\xi_3 = -V$.

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MOTION OF A HEAT-CONDUCTING GAS ACTED ON BY A HEAT-INSULATED EXPANDING PISTON

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The temperature and density fields associated with the motion of an ideal gas acted on by an expanding piston have singularities at the piston surface [1-3]. These arise through nonallowance for heat conduction by the gas, which plays the determining role near the surface of the piston.

We shall solve the problem of motion of a heat-conducting gas acted on by an expanding heat-insulated piston by the method of interior and exterior expansions. To this end we construct the principal term of the interior asymptotic expansion by splicing it with the solution for an ideal gas which constitutes the principal term of the exterior asymptotic expansion. This yields a solution free from singularities. A similar solution for the strong-detonation problem was obtained by Sychev [4].

1. Let a plane, cylindrical, or spherical piston expand according to the law

$$y = At^k \qquad (A,\ k = \text{const})$$

in an undisturbed gas of density $\rho_0 = \text{const}$.

We assume that the gas is viscous and heat-conducting, and that its viscosity μ is related to the enthalpy h by the expression

$$\mu = Ch^m\ (C, m = \text{const})$$

The Prandtl number $\sigma = \text{const}$. Neglecting counterpressure and assuming that the surface of the piston is heat-insulated, we infer that the determining parameters are ρ_0, C, A. The determining parameters, the time t, and the space coordinate y can be combined into two dimensionless variables,

$$\xi_1 = \frac{y}{At^k}, \qquad \xi_2 = \frac{CA^{2(m-1)}}{\rho_0 t^\alpha} \qquad (\alpha = 1 - 2(k-1)(m-1) \tag{1.1}$$

From (1.1) we infer that for $\alpha > 0$ the effect of viscosity and heat conduction on gas flow diminishes with time; for $\alpha = 0$ the problem becomes self-similar even with allowance for viscosity and heat conduction, and for $\alpha < 0$ the effect of viscosity and heat conduction increases with time.

From now on we shall consider the case $\alpha > 0$, which is of the greatest interest and corresponds to the real values

$$1/_2 \leqslant m \leqslant 1,\ k > 2 / (\nu + 3)$$

where $\nu = 0,1,2$, respectively, for the plane, cylindrical, and spherical cases.

We can combine the determining parameters into quantities having the dimensions of time and length which we shall use as our scales,

$$t_1 = [C\rho_0^{-1}A^{2(m-1)}]^{1/\alpha}, \qquad l_1 = [C^k\rho_0^{-k}A^{2m-1}]^{1/\alpha} \tag{1.2}$$

Denoting the velocity by v and the pressure by p, we introduce dimensionless values for the independent and dependent variables,

$$t^\circ = tt_1^{-1}, \qquad y^\circ = yl_1^{-1}, \qquad v^\circ = vt_1l_1^{-1}, \tag{1.3}$$

$$p^\circ = p\rho_0^{-1}t_1^2l_1^{-2}, \qquad \rho^\circ = \rho\rho_0^{-1}, \qquad h^\circ = ht_1^2l_1^{-2}$$

Let us write the Navier-Stokes equation for a one-dimensional viscous heat-conducting gas in dimensionless parameters (the mark $^\circ$ identifying dimensionless quantities will be omitted for simplicity),

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left[h^m\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\nu\frac{v}{y}\right)\right] + 2\nu\frac{h^m}{y}\left(\frac{\partial v}{\partial y} - \frac{v}{y}\right)$$

$$\rho\left(\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial y}\right) = \frac{\partial p}{\partial t} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}y^{-\nu}\frac{\partial}{\partial y}\left(y^\nu h^m\frac{\partial h}{\partial y}\right) +$$
$$+ 2h^m\left[\left(\frac{\partial v}{\partial y}\right)^2 + \nu\left(\frac{v}{y}\right)^2\right] - \frac{2}{3}h^m\left(\frac{\partial v}{\partial y} + \nu\frac{v}{y}\right)^2$$

$$\frac{\partial}{\partial t}(\rho y^\nu) + \frac{\partial}{\partial y}(\rho v y^\nu) = 0, \qquad p = \frac{\gamma - 1}{\gamma}\rho h, \qquad \gamma = \frac{c_p}{c_v} \tag{1.4}$$

Let us convert from the Eulerian variables y, t to the Lagrangian variables ψ, t. By virtue of the continuity equation these are related by the expressions

$$\left.\frac{\partial}{\partial t}\right|_{y=\mathrm{const}} = \left.\frac{\partial}{\partial t}\right|_{\psi=\mathrm{const}} - \rho v y^\nu \frac{\partial}{\partial \psi}, \qquad \frac{\partial}{\partial y} = \rho y^\nu \frac{\partial}{\partial \psi} \tag{1.5}$$

System (1.4) now becomes

$$\rho\frac{\partial v}{\partial t} + \rho y^\nu \frac{\partial p}{\partial \psi} = \rho y^\nu \frac{\partial}{\partial \psi}\left[h^m\left(\frac{4}{3}\rho y^\nu \frac{\partial v}{\partial \psi} - \frac{2}{3}\nu\frac{v}{y}\right)\right] + {} + 2\nu\frac{h^m}{y}\left(\rho y^\nu \frac{\partial v}{\partial \psi} - \frac{v}{y}\right)$$

$$\rho\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} + \frac{\rho}{\sigma}\frac{\partial}{\partial \psi}\left(y^{2\nu}\rho h^m\frac{\partial h}{\partial \psi}\right) + 2h^m\left[\left(\rho y^\nu\frac{\partial v}{\partial \psi}\right)^2 + {} + \nu\left(\frac{v}{y}\right)^2\right] - \frac{2}{3}h^m\left(\rho y^\nu \frac{\partial v}{\partial \psi} + \nu\frac{v}{y}\right)^2$$

$$\rho y^\nu \frac{\partial y}{\partial \psi} = 1, \qquad \frac{\partial y}{\partial t} = v, \qquad p = \frac{\gamma - 1}{\gamma}\rho h \tag{1.6}$$

Now let us find the principal terms of the asymptotic expansions of the solution of system (1.6) as $t \to \infty$ which satisfy the initial and boundary conditions.

2. We shall attempt to find the exterior asymptotic expansion in the form

$$y = a_0 t^k [Y_0(n) + O(t^{-\alpha})], \qquad v = \frac{2k}{\gamma+1} a_0 t^{k-1}[V_0(n) + O(t^{-\alpha})]$$

$$p = \frac{2k^2}{\gamma+1} a_0^2 t^{2(k-1)}[P_0(n) + O(t^{-\alpha})], \qquad \rho = \frac{\gamma+1}{\gamma-1}[R_0(n) + O(t^{-\alpha})] \tag{2.1}$$

$$h = \frac{2\gamma k^2}{(\gamma+1)^2} a_0^2 t^{2(k-1)}[H_0(n) + O(t^{-\alpha})]$$

$$n = (\nu+1)\, a_0^{-(1+\nu)} t^{-(1+\nu)k}\psi, \qquad a_0 = \mathrm{const}$$

Substituting (2.1) into (1.6) and collecting the principal terms of the equations, we obtain a system for determining the functions $Y_0(n), V_0(n), \ldots, H_0(n)$,

$$(k-1)V_0 - (1+\nu)knV_0' + (1+\nu)kY_0^\nu P_0' = 0, \qquad (n^{\beta\gamma}P_0R_0^{-\gamma})' = 0$$

$$(1+\nu)\frac{\gamma+1}{\gamma-1}R_0Y_0^\nu Y_0' = 1, \qquad Y_0 - (1+\nu)nY_0' = \frac{2}{\gamma+1}V_0 \tag{2.2}$$

$$P_0 = R_0H_0 \qquad \left(\beta = \frac{2(1-k)}{k(1+\nu)\gamma}\right)$$

(the prime indicates the derivative with respect to n).

The required functions satisfy the following conditions at the shock wave:

$$Y_0(1) = V_0(1) = P_0(1) = R_0(1) = H_0(1) = 1 \tag{2.3}$$

System (2.2) with boundary conditions (2.3) describes the self-similar flow of an ideal gas acted on by an expanding piston.

The behavior of the solution of system (2.2) near the piston surface ($n = 0$) is described by the relations

$$
\begin{gathered}
Y_0 = Y_{00} + Y_{01} n^{1-\beta}\,[1 + O(n^s)], \quad V_0 = V_{00} + V_{01} n^{1-\beta}\,[1 + O(n^s)] \\
H_0 = H_{00} n^{-\beta}\,[1 + O(n)], \quad R_0 = R_{00} n^{\beta}\,[1 + O(n)] \\
P_0 = P_{00} + P_{01} n [1 + O(n^{1-\beta})] \qquad (2.4) \\
s = 1 - \beta \quad \text{for } \beta > 0, \qquad s = 1 \quad \text{for } \beta < 0 \\
V_{00} = \frac{\gamma+1}{2} Y_{00}, \quad R_{00} = P_{00}^{1/\gamma}, \quad H_{00} = P_{00}^{\frac{\gamma-1}{\gamma}} \\
Y_{01} = \frac{\gamma-1}{(\gamma+1)(1+\nu)(1-\beta)} Y_{00}^{-\nu} P_{00}^{-1/\gamma}, \quad P_{01} = \frac{(\gamma+1)(1-k)}{2k(1+\nu)} Y_{00}^{1-\nu} \\
V_{01} = \frac{\gamma-1}{2}\,[(1+\nu)^{-1}(1-\beta)^{-1} - 1]\, Y_{00}^{-\nu} P_{00}^{-1/\gamma}, \quad a_0 = Y_{00}^{-1}
\end{gathered}
$$

The constants P_{00}, Y_{00} can be found from the complete solution of system (2.2) with boundary conditions (2.3) of [3].

3. To find the interior asymptotic expansion in the interior flow region we introduce the quantity $N = nt^{\delta}$, where $\delta > 0$, as our independent variable of order unity. Making use of the principle of splicing interior and exterior expansions [4,5], we can express the limits of the exterior expansion in terms of the variables of the interior expansion

$$
\begin{gathered}
y = a_0 t^k\,[Y_{00} + Y_{01} N^{1-\beta}\, t^{-\delta(1-\beta)}\,(1 + O(t^{-\delta s}))] \\
v = \frac{2k}{\gamma+1} a_0 t^{k-1}\,[V_{00} + V_{01} N^{1-\beta}\, t^{-\delta(1-\beta)}\,(1 + O(t^{-\delta s}))] \\
p = \frac{2}{\gamma+1} k^2 a_0^2 t^{2(k-1)}\,[P_{00} + P_{01} N t^{-\delta}\,(1 + O(t^{-\delta(1-\beta)}))] \qquad (3.1) \\
\rho = \frac{\gamma+1}{\gamma-1} R_{00} N^{\beta}\, t^{-\delta\beta}\,[1 + O(t^{-\delta})] \\
h = \frac{2\gamma k^2}{(\gamma+1)^2} a_0^2 t^{2(k-1)} H_{00} N^{-\beta}\, t^{\delta\beta}\,[1 + O(t^{-\delta})]
\end{gathered}
$$

We determine δ from the condition that the interior region is that neighborhood of the piston surface in which heat conduction plays the determining role. The energy equation then implies that

$$
\delta = \frac{\alpha}{2 + \beta(m-1)} > 0
$$

From (3.1) we infer that the interior asymptotic expansion must be found in the form

$$
\begin{gathered}
y = a_0 t^k\,[y_0(N) + y_1(N)\, t^{-\delta(1-\beta)}\,(1 + O(t^{-\delta s}))] \\
v = \frac{2k}{\gamma+1} a_0 t^{k-1}\,[v_0(N) + v_1(N)\, t^{-\delta(1-\beta)}\,(1 + O(t^{-\delta s}))] \\
p = \frac{2k^2}{\gamma+1} a_0^2 t^{2(k-1)}\,[p_0(N) + p_1(N)\, t^{-\delta}\,(1 + O(t^{-\delta(1-\beta)}))] \\
\rho = \frac{\gamma+1}{\gamma-1} t^{-\delta\beta} \rho_0(N)\,[1 + O(t^{-\delta})] \\
h = \frac{2\gamma k^2}{(\gamma+1)^2} a_0^2 t^{2(k-1)+\delta\beta} h_0(N)\,[1 + O(t^{-\delta})] \qquad (3.2)
\end{gathered}
$$

From the condition of splicing of expansions (3.2) with the exterior expansion we find that

$$y_0(N) \to Y_{00}, \quad y_1(N) \to Y_{01} N^{1-\beta}, \quad v_0(N) \to V_{00}$$
$$v_1(N) \to V_{01}N^{1-\beta}, \quad \rho_0(N) \to R_{00}N^{\beta}, \quad p_0(N) \to P_{00} \tag{3.3}$$
$$p_1(N) \to P_{01}N, \quad h_0(N) \to H_{00}N^{-\beta}$$

as $N \to \infty$.

Substituting expansions (3.2) into (1.6) and combining terms containing equal powers of t, we obtain a system of equations for determining the coefficients of the internal expansion

$$y_0' = 0, \quad y_0 = \frac{2}{\gamma+1} v_0, \quad p_0' = 0, \quad p_0 = \rho_0 h_0$$
$$\beta h_0 + N h_0' + B y_0^{2\nu} p_0 (h_0^{m-1} h_0')' = 0$$
$$(1+\nu)\frac{\gamma+1}{\gamma-1}\rho_0 y_0^{\nu} y_1' = 1$$
$$[k - \delta(1-\beta)]\, y_1 + [\delta - k(1+\nu)]\, N y_1' = \frac{2k}{\gamma+1} v_1$$
$$(k-1)\, v_0 + y_0^{\nu} k (1+\nu)\, p_1' = 0 \tag{3.4}$$
$$B = \frac{1}{\sigma} a_0^{2(m+1)} k^{2m} (1+\nu)^2 \left[\frac{2\gamma}{(\gamma+1)^2}\right]^m \frac{(\gamma+1)[2+\beta(m-1)]}{(\gamma-1)[2(1+\nu)k-1]} > 0$$

Solving system (3.4) with boundary conditions (3.3) and the conditions

$$h_0' = 0, \quad y_1 = 0, \quad v_1 = 0$$

at the piston surface $(N = 0)$, we obtain

$$y_0 = Y_{00}, \quad v_0 = V_{00}, \quad p_0 = P_{00}, \quad \rho_0 = P_{00} h_0^{-1} \tag{3.5}$$

The function $h_0(N)$ can be determined from the fifth equation of system (3.4) and the boundary conditions

$$h_0'(0) = 0, \quad h_0(N) \to H_{00}N^{-\beta} \quad \text{as} \quad N \to \infty$$

The invariant transformation

$$h_0 \to C_1 h_0, \qquad N \to C_1^{(m-1)/2} N \tag{3.6}$$

of the equation and boundary condition for $N = 0$ reduces the boundary value problem for $h_0(N)$ to the Cauchy problem in which $h_0(0)$ and $h_0'(0) = 0$. Moreover, on making the substitution

$$N = N_1 (B Y_{00}^{2\nu} P_{00})^{1/2} \tag{3.7}$$

in the fifth equation of system (3.4), we obtain the equation

$$(h_0^{m-1} h_0')' + N_1 h_0' + \beta h_0 = 0 \tag{3.8}$$

If we have a system of integral curves of Eq. (3.8) which satisfy the initial conditions $h_0(0) = 1, h_0'(0) = 0$ for various values of m and β, ($1/_2 \leqslant m \leqslant 1, -2 < \beta < 1$), then we can use substitution (3.7) and invariant transformation (3.6) to obtain solutions of the fifth equation of system (3.4) for various values $1/_2 \leqslant m \leqslant 1$, $k > 2/(\nu + 3)$, $\gamma > 1$, $\nu = 0,1,2$, satisfying the appropriate conditions of splicing of the exterior and interior expansions.

We note that Eq. (5.4) derived in [4] with the initial condition for $N = 0$ has a similar invariant transformation, which means that the solution of the boundary value problem also reduces to the solution of a Cauchy problem.

Solving the remaining equations of system (3.4), we obtain

$$y_1 = \frac{\gamma - 1}{(\gamma + 1)(1 + \nu)(1 - \beta)} Y_{00}^{-\nu} P_{00}^{-1} (N h_0 + B Y_{00}^{2\nu} P_{00} h_0^{m-1} h_0')$$

$$v_1 = \frac{\gamma - 1}{2(1 + \nu)(1 - \beta)} P_{00}^{-1} Y_{00}^{-\nu} \{[1 - (1 - \beta)(1 + \nu)] N h_0 +$$

$$+ [1 - \delta (1 - \beta) k^{-1}] B Y_{00}^{2\nu} P_{00} h_0^{m-1} h_0'\}$$

$$p_1 = P_{01} N + C_2 \qquad (C_2 = \text{const}) \tag{3.9}$$

The density and enthalpy at the piston surface are given by the relations

$$\frac{\rho(0)}{\rho_2} = P_{00} h_0^{-1}(0)\, t^{-\delta\beta}, \qquad \frac{h(0)}{h_2} = h_0(0)\, t^{\delta\beta}$$

where ρ_2 and h_2 are the density and enthalpy at the shock wave surface.

4. In the particular case where $\beta = -1$, $m = 1$, which corresponds to $k = 2/(2 - \gamma)$ in the plane case, it is possible to obtain the exact solution of the fifth equation of system (3.4) with the boundary conditions

$$h_0 = H_{00} \left(\frac{2}{\pi b}\right)^{1/2} \left[e^{-1/_2 bN^2} + bN \int_0^N e^{-1/_2 bN^2} dN\right]$$

$$b = (B Y_{00}^{2\nu} P_{00})^{-1} \tag{4.1}$$

Now, substituting (4.1) into (3.9), we obtain (4.2)

$$y_1 = \frac{\gamma - 1}{(\gamma + 1)(1 + \nu)(1 - \beta)} Y_{00}^{-\nu} P_{00}^{-1/\gamma} \left(\frac{2}{\pi b}\right)^{1/2} \left[(bN^2 - 1) \int_0^N e^{-1/_2 bN^2} dN + N e^{-1/_2 bN^2}\right]$$

Formulas (4.1) and (4.2) describe the enthalpy field in the neighborhood of the piston surface. The temperature at the piston surface is here defined by the relation

$$T = T_2 H_{00} \left(\frac{2}{\pi b}\right)^{1/2} t^{-1/_2}.$$

The quantity H_{00} can be expressed in terms of the P_{00} which is determined by solving the corresponding problem for an ideal gas.

In the more general case where $m = 1$ and β is arbitrary the solution can be expressed in terms of the degenerate hypergeometric functions $\Phi(a, b, z)$

$$h_0 = C_3 \Phi\left(\frac{\beta}{2},\ \frac{1}{2},\ -\frac{N_1^2}{2}\right)$$

$$y_1 = \frac{\gamma - 1}{(\gamma + 1)(1 + \nu)(1 - \beta)} C_3 Y_{00}^{-\nu} P_{00}^{-1} \left[\Phi\left(\frac{\beta}{2},\ \frac{1}{2},\ -\frac{N_1^2}{2}\right) - \right.$$
$$\left. - \beta\Phi\left(\frac{\beta}{2} + 1,\ \frac{3}{2},\ -\frac{N_1^2}{2}\right)\right] N$$

$$v_1 = \frac{\gamma + 1}{2(1 + \nu)(1 - \beta)} C_3 P_{00}^{-1} Y_{00}^{-\nu} \left\{[1 - (1 - \beta)(1 + \nu)]\Phi\left(\frac{\beta}{2},\ \frac{1}{2},\ -\frac{N_1^2}{2}\right) + \right.$$
$$\left. + [1 - \delta(1 - \beta) k^{-1}]\beta\Phi\left(\frac{\beta}{2} + 1,\ \frac{3}{2},\ -\frac{N_1^2}{2}\right)\right\} N \qquad (4.3)$$

$$C_3 = H_{00} \frac{\Gamma(1/2 - 1/2\beta)}{\Gamma(1/2)\, 2^{1/2\beta}} (B Y_{00}^{2\nu} P_{00})^{-1/2\beta} = \mathrm{const}$$

Here N_1 is given by relation (3.7) and $\Gamma(z)$ is a gamma function. For $\beta = -1$ relations (4.3) become (4.1), (4.2).

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